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THE CALCULATION OF THE NATURAL FREQUENCY OF A CANTILEVER MONOPLANE WING

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THE CALCULATION OF THE NATIONAL FREQUENCY OF A CANTILEVER MONOPLANE WING

Prepared by Stanley R. Carpenter, Materiel Division, Air Corps, Wright Field, Dayton, Ohio, December 2, 1929

SUMMARY

The purpose of this report is to present a practical application of the calculation of the natural frequency of a stressed skin monoplane wing in bending and in torsion. The theories of the methods employed may be found in the following papers:

Preliminary Study of Fatigue Failures of Metal Propellers Caused by Engine Impulses and Vibration, Air Corps Information Circular, Volume VII, No. 618, by John E. Younger.

Simple Approximate Method of Determining the Natural Frequency of Torsional Vibration (With Particular Reference to Monoplane Wings and Propeller Blades), Airplane Department Memorandum No. 1062, Matériel Division, Air Corps, Wright Field, by John E. Younger.

The calculations are applied to the Fokker C-2A monoplane wing, the final results of which are compared to the experimental results reported in the following paper:

Determination of the Elastic Axis and Natural Periods of Vibration of the C-2A Monoplane Wing, Airplane Department Memorandum No. 1066, Matériel Division, Air Corps, Wright Field, by Charles J. Spere.

The comparison of the experimental and calculated results is as follows:

	Experimental	Calculated	Per cent of error
Frequency in bending	3.95	4.37	10.60
Frequency in torsion	12.00	9.83	18.10

In the Air Corps Information Circular No. 618 mentioned above, the natural frequency of vibration in bending is given as

$$f_b=\frac{\sqrt{g}}{2\pi}\sqrt{\frac{\Sigma W_1Y_1+W_2Y_2+\cdots\cdots W_nY_n}{\Sigma W_1Y_1^2+W_2Y_2^2+\cdots\cdots W_nY_n^2}}\quad(1)$$

in which—

f_b = Natural frequency of bending in complete cycles per second.

W_n = Weight per running length of span at different intervals.

Y_n = Corresponding deflections due to W_n.

The computations for the deflections Y are based upon the relationship between the loading, shear, moment, slope, and deflection curves expressed in the following:

$$\text{Loading} = w \quad(2)$$

$$\text{Shear} = \int_0^L w\,dx \quad(3)$$

$$\text{Moment} = \int_0^L (\text{Shear})\,dx \quad(4)$$

$$\text{Slope} = \frac{1}{E}\int_0^L \frac{(\text{Moment})\,dx}{I} \quad(5)$$

$$\text{Deflection} = \int_0^L (\text{Slope})\,dx \quad(6)$$

In all the following computations, the graphical method of integration was used in finding the results of the above formulæ. Each curve was plotted against semispan. The semispan was divided into a number of small equal sections from which mean values were found and summed in deriving one curve from another.

The loading curve w was computed from the weight summation of the various component parts of the wing (ribs, spars, and plywood covering) plus a correction factor for each section to make the computed weight of the wing equal to the actual weight. A correction factor is usually necessary because of errors in assumptions and neglect of weight of glue, paint, varnish, nails, fittings, etc.

The weight of the semispan from the computed loading curve, AB, Figure 1, is equal to the area under the curve multiplied by the scale to which it is drawn and is approximately

$$\frac{2.8\times460}{2}=644 \text{ pounds. (See Figure 1.)}$$

The actual weight of the semispan $=\frac{1730.25}{2}=865$ pounds.

Difference of actual and computed weight $=865-644=221$ lb./semispan.

This weight difference was divided equally into 10 parts and assumed to be distributed over 10 equal areas in the semispan.

A = area of semispan = 373.3 square feet.

$\triangle A=\frac{373.3}{10}=37.33$ square feet = 5,380 square inches.

$\triangle$ weight $=\frac{220}{10}=22$ pounds/$\triangle A$ uniformly distributed.

$\triangle A$ (equal to $\triangle x$ times mean chord) was found by scaling from the wing drawing, values of $\triangle x$, the increment of span, and the mean chord to give $\triangle A$ equal 5,380 square inches.

$$\frac{\triangle wt}{\triangle x}=\frac{22}{\triangle x}$$

will give the correction factors in pounds per unit length to be applied over increment $\triangle x$. The values thus found are given in the following table:

Area	1	2	3	4	5
Mean chord, inches	149½	149½	145	140	134
$\triangle x$, inches	36	36	37	38½	40
Correction, pounds per inch	0.61	0.61	0.60	0.57	0.55

Area	6	7	8	9	10
Mean chord, inches	130	125	114	103	81½
$\triangle x$, inches	41½	43	47	52	66
Correction, pounds per inch	0.53	0.51	0.47	0.43	0.39

The corrections were plotted against semi-span in Figure 2 from which values for every 20 inches of span were added to AB to find the loading curve.

This method of finding the loading curve is somewhat different from that used by the United States Army. The Army method is to find—

w=weight per square foot

$$=\frac{\text{weight of wing}}{\text{area}}=\frac{865}{375}=2.3 \text{ pounds per square foot,}$$

so that the weight per unit length of wing=w times mean chord.

The sections were divided into 2-foot lengths. For section 9, the mean chord is 10.8 feet, so that the weight per running foot of wing is—

$$10.8\times 2.3=24.9 \text{ pounds,}$$

and the weight per running inch of wing is—

$$\frac{24.9}{12}=2.07 \text{ pounds.}$$

The weight per running inch at the various sections is given in the following table:

Section	1	2	3	4	5	6
Mean chord	12.50	12.50	12.50	12.50	12.25	11.90
Wt./inch	2.40	2.40	2.40	2.40	2.35	2.28

Section	7	8	9	10	11	12
Mean chord	11.50	11.12	10.80	10.50	10.12	9.75
Wt./inch	2.21	2.13	2.07	2.02	1.94	1.87

Section	13	14	15	16	17	18
Mean chord	9.37	9.00	8.65	8.25	8.00	6.75
Wt./inch	1.80	1.73	1.66	1.58	1.53	1.29

This table is plotted in Figure 3 with the loading curve for the purpose of comparison. The former method is more accurate.

The semi-span was divided into small sections for the graphical integration as designated in column 1, page 7, and the distance between each section is given in column 2. Column 3 gives the mean weight of wing per inch for each section taken from the upper curve of Figure 1.

The mean results at any section (designated by two station numbers) are indicated on the horizontal line between the two stations of that section. Results on the same line as the stations represent the values there and are computed from the mean quantities for that section.

Thus, column 4 gives the weight at each station for the wing section to the right (if the right half of the span is considered) and,

Weight of section at station 19=mean w for section 19–20 multiplied by $dx=0.94\times 20=18.8$ pounds.

Column 5 is the evaluation of equation (3) representing the total vertical shear at any points, and is the summation of the shear at all the stations above that point in column 4, since the vertical shear (zero at tip) increases to a maximum at the root. Thus, for station 19,

Vertical shear=18.8+16.6+9.2=44.6 pounds, and the mean shear between 19 and $20=\frac{44.6+25.8}{2}=35.2$ pounds.

These values are used to find the results in column 7, which is the mean shear times dx.

The total moment given by equation (4),

$$M=\int_0^L (\text{Shear})\ dx$$

is represented in column 8, the summation of mean shear times dx. For station 19,

total moment=Σ mean shear times dx of all the sections between station 19 and the tip.
=704+350+78+0=1132 pound-inches.

From the moment the slope is found by equation (5).

$$\text{Slope}=\frac{1}{E}\int_0^L \frac{(\text{Moment})\ dx}{I}$$

$$=\frac{1}{E}\Sigma\frac{M}{I}dx$$

in which—

I=Mean moment of inertia in inches^4 of both spars and part of the skin for each section.
M=Corresponding mean moment in pound-inches.
dx=Length of section in inches.
E=Modulus of elasticity in pounds per square inch.

In calculating I, the plywood covering approximately four times the width of spar on the top and bottom of both spars was considered as part of the spar. Figure 4 gives the values thus computed, which are listed in column 10.

Column 11 is the mean values of $\frac{M}{I}dx$ and for any station in column 12 the value is for the summation of all the sections below that station in column 11. $\Sigma\frac{M}{I}dx$ for any station in column 12, when divided by E, will give the slope at that point.

The slope at station 3, column 12,

$$= \frac{1}{E} \Sigma \frac{M}{I}\, dx$$

$$= \frac{1}{E} (321 + 287 + 258)$$

$$= \frac{1}{E}\, 866$$

The value E is not substituted until the deflection is found in column 16.

Column 14 is the summation of the average $\frac{M}{I}\, dx$ in column 13, starting at the root, because the slope there is zero.

Column 15 is the values of column 14 multiplied by dx. In column 16, E (1,300,000 pounds per square inch) is substituted to find the deflection of the wing in inches due to its own weight. The average deflection in feet for each section is given in column 18.

The mean values of WY and WY^2 are listed in columns 20 and 21, respectively, the summation of which is required for formula (1),

$$f_b = \frac{\sqrt{g}}{2\pi} \sqrt{\frac{\Sigma W_n Y_n}{\Sigma W_n Y_n^2}}$$

$$\Sigma W_n Y_n = 0.7622$$

$$\Sigma W_n Y_n^2 = 0.0324$$

$$f_b = \frac{\sqrt{32.2}}{2\pi} \sqrt{\frac{0.7622}{0.0324}} = 0.905 \times 4.85$$

$$= 4.37 \text{ cycles per second.}$$

The mean experimental value of $f = 3.95$ cycles per second.

Per cent of error $= \frac{4.37 - 3.95}{3.95} \times 100 = 10.6$ per cent.

The shear, moment, slope, and deflection curves are plotted in Figures 5, 6, 7, and 8. The shear and moment curves are negative; they are zero at the tip of the wing and increase to a maximum at the center. The slope and deflection curves are also negative but are zero at the center with their maximum value at the tip.

Great care must be exercised with the units. The mean deflections Y, in column 18, must be in feet for the substitution in formula (1). It is usually easier to work in inch units as in these computations. If inch units are employed, use—

E in pounds per square inch.
I in inches ^{4}.
dx in inches.
W in pounds per running inch.
M in pound-inches.

Y will then be in inches, which may be readily converted to feet for substitution in the formula.

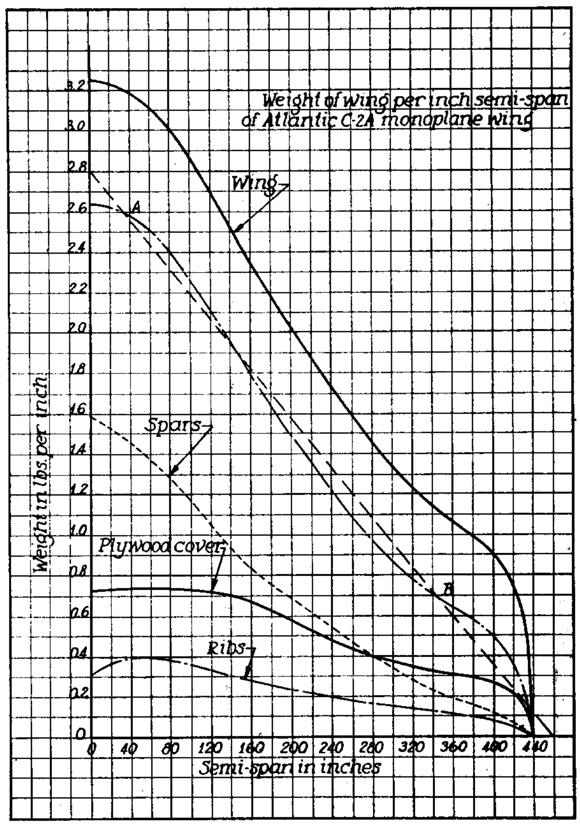

FIGURE 1

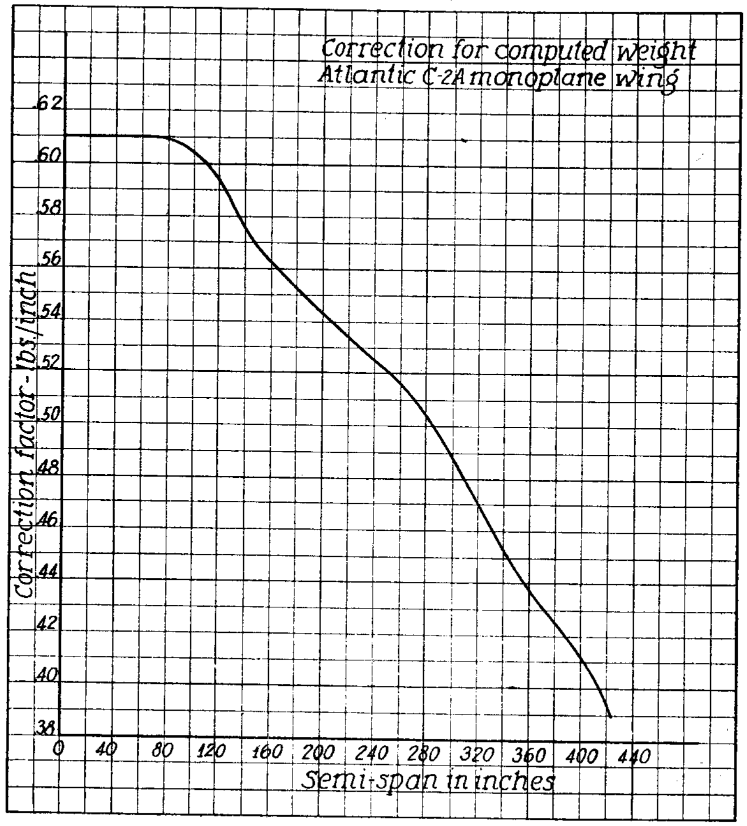

FIGURE 2

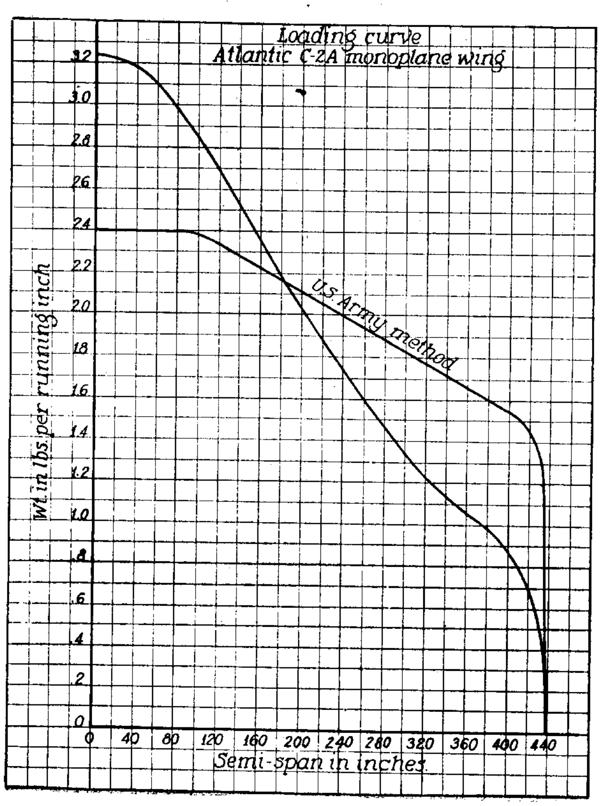

FIGURE 3

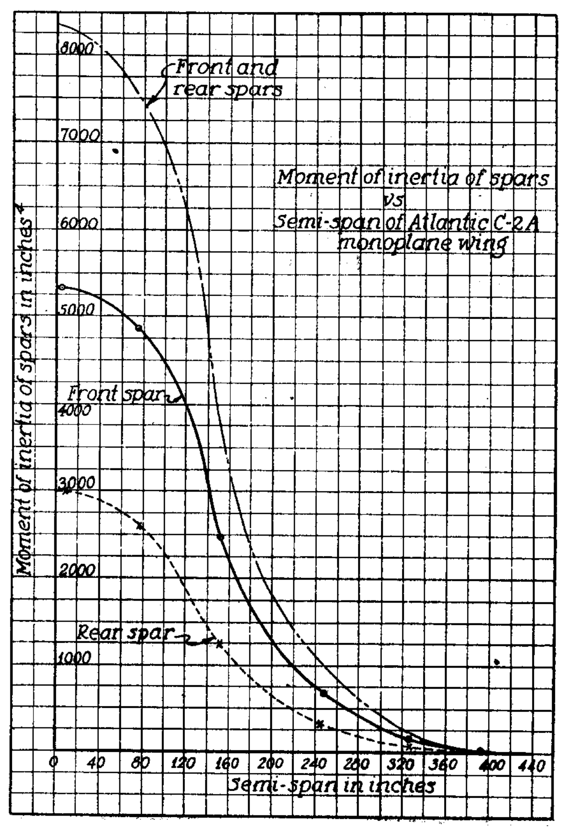

FIGURE 4

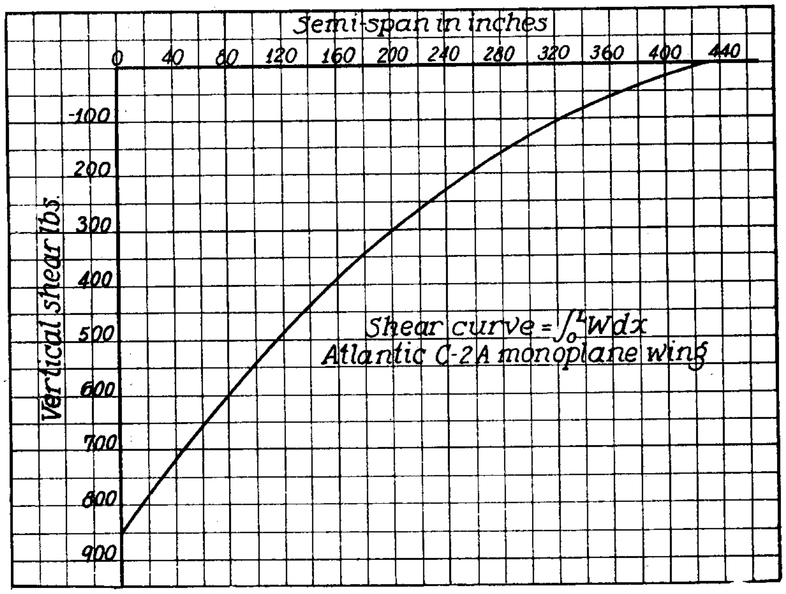

FIGURE 5

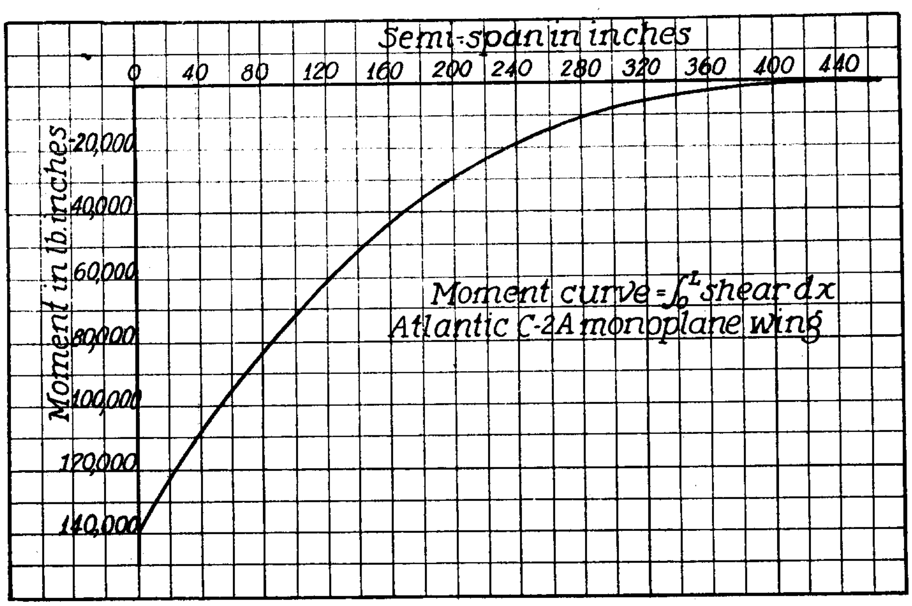

FIGURE 6

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Station	dx	W	Wdx	Shear= Σ Wdx	Mean shear	Mean shear dx	Moment = Σ mean shear dx	M = Mean moment	I	$\frac{M}{I}dx$	$\Sigma \frac{M}{I}dx$	Mean $\Sigma \frac{M}{I}dx$	$\frac{\Sigma \text{ Mean slope}}{E}$	$\frac{\Sigma \text{ Mean slope dx}}{E}$	Deflection = $\frac{\Sigma \text{ Mean slope dx}}{1{,}300{,}000}$	Deflection	Y = Mean deflection	Y^2	WY	WY^2
	Inches	*Lbs. per in.*	*Lbs.*	*Lbs.*	*Lbs.*	*Lbs. in.*	*Lbs. in.*	*Lbs. in.*	*Inches*							*Feet*	*Feet*			
Tip			0	0		0	0				7,623			1,450,820	1.11	0.0929				
	17	0.54			4.6			39	5	133		7,556	72,541				0.0858	0.00735	0.0464	0.00396
21			9.2	9.2		78	78				7,490			1,299,700	1.00	.0788				
	20	.83			17.5			253	10	506		7,237	64,985				.0764	.00582	.0634	.00483
20			16.6	25.8		350	428				6,984			1,154,960	.885	.0740				
	20	.94			35.2			780	25	624		6,672	57,748				.0698	.00486	.0656	.00456
19			18.8	44.6		704	1,132				6,360			1,021,520	.785	.0655				
	20	1.02			54.8			1,680	60	560		6,080	51,076				.0615	.00379	.0626	.00386
18			20.4	65.0		1,096	2,228				5,800			899,920	.691	.0576				
	20	1.09			75.9			2,987	100	597		5,502	44,996				.0541	.00292	.0590	.00318
17			21.8	86.8		1,518	3,746				5,204			789,880	.606	.0505				
	20	1.18			98.6			4,732	200	473		4,968	39,494				.0473	.00224	.0559	.00264
16			23.6	110.4		1,972	5,718				4,731			690,520	.530	.0442				
	20	1.28			123.2			6,950	350	397		4,532	34,526				.0413	.00171	.0528	.00219
15			25.6	136.0		2,464	8,182				4,334			599,880	.462	.0384				
	20	1.40			150.0			9,682	500	386		4,141	29,994				.0358	.00128	.0501	.00179
14			28.0	164.0		3,000	11,182				3,948			517,060	.398	.0331				
	20	1.52			179.2			12,974	725	358		3,769	25,853				.0307	.00094	.0466	.00143
13			30.4	194.4		3,584	14,766				3,590			441,680	.339	.0282				
	20	1.66			211.0			16,876	950	356		3,412	22,084				.0260	.000675	.0431	.00112
12			33.2	227.6		4,220	18,986				3,234			373,440	.287	.0239				
	20	1.81			246.0			21,446	1,250	344		3,062	18,672				.0220	.000485	.0402	.000878
11			36.2	263.8		4,920	23,906				2,890			312,200	.240	.0200				
	20	1.95			283.5			26,741	1,600	334		2,723	15,610				.01825	.000333	.0356	.00065
10			39.0	302.8		5,670	29,576				2,556			257,740	.198	.0165				
	20	2.11			324.0			32,816	2,050	320		2,396	12,887				.01495	.000224	.0316	.000474
9			42.2	345.0		6,480	36,056				2,236			209,820	.161	.0134				
	20	2.26			369.0			39,746	2,750	289		2,092	10,491				.01208	.000146	.0273	.00033
8			45.2	390.2		7,380	43,436				1,947			167,980	.129	.01075				
	20	2.44			415.0			47,586	4,000	238		1,828	8,399				.00959	.000092	.0234	.000224
7			48.8	439.0		8,300	51,736				1,709			131,420	.101	.00842				
	20	2.61			465.0			56,386	5,700	198		1,610	6,571				.00739	.0000546	.0193	.000143
6			52.2	491.2		9,300	61,036				1,511			99,220	.076	.00635				
	20	2.77			519.0			66,226	6,650	199		1,412	4,961				.00545	.0000297	.0151	.0000824
5			55.4	546.6		10,380	71,416				1,312			70,980	.056	.00454				
	20	2.92			576.0			77,173	7,250	213		1,205	3,549				.00377	.0000142	.0110	.0000415
4			58.4	605.0		11,520	82,936				1,099			46,880	.036	.00300				
	20	3.06			636.0			89,296	7,650	233		983	2,344				.00237	.0000056	.00725	.0000171
3			61.2	666.2		12,720	95,656				866			27,220	.021	.00174				
	20	3.15			698.0			102,636	7,950	258		737	1,361				.00127	.00000161	.00400	.00000507
2			63.0	729.2		13,960	109,616				608			12,480	.001	.0008				
	20	3.21			761.0			117,226	8,175	287		464	624				.000502	.000000252	.00161	.00000081
1			64.2	793.4		15,220	124,836				321			3,200		.000205				
	20	3.22			826.0			133,096	8,300	321		160	160				.0001025	.00000001025	.00033	.00000033
Root			64.4	858.0		16,520	141,356			0	0			0	0	0				

$\Sigma W_n Y_n = .7622$ $\Sigma W_n Y_n^2 = .03241$

CALCULATION OF FREQUENCY IN TORSION

The natural frequency of vibration in torsion of a monoplane wing is shown in A. D. M. 1062 to be—

$$f_t=\frac{1}{2\pi}\sqrt{\frac{\Sigma J_m\theta_o}{\Sigma J_m\theta^2_o}} \text{ - - - - - - - - - - - - - - - - (7)}$$

where—

f_t = The natural frequency of vibration in complete cycles per second.

J_m = The mass polar moment of inertia per unit length for each wing section measured in slug inches squared.

θ_o = The total angle of twist of the shell in radians between the root and the section under consideration when subjected to a distributed torsional moment.

$=\Delta\theta_1+\Delta\theta_2+\Delta\theta_3+\text{ - - - - - - }\Delta\theta_n.$

$\Delta\theta_1$, $\Delta\theta_2$, $\Delta\theta_3$, etc., are increments of twist at the various sections.

The equation of θ_o given in A. D. M. 1058 is

$$\theta_o=\frac{QLdx}{4A^2tE_s} \text{ - - - - - - - - - - - - - - - - - (8)}$$

in which—

$Q=\Sigma J_m$.

L = The mean length of periphery of the section in inches.

J_m has the same meaning as in equation (7).

dx = The length of section in inches subjected to torque.

t = The mean thickness of stressed cover in inches.

A = The average area in square inches bounded by periphery, L.

E_s = The modulus of elasticity of the cover in shear in pounds per square inch.

With reference to A. D. M. 1059, the total area of the airfoil section, A, is—

$$A=0.725\ hc \text{ to } 0.785\ hc \text{ - - - - - - - - - - (9)}$$

For these computations the area of the airfoil is taken—

$$A=0.731\ hc.$$

$$L=\left[2.7\left(\frac{h}{c}\right)^2+2\right]c \text{ - - - - - - - - - - - - - - (10)}$$

$$I_{\bar{x}}=(0.119h+0.256c)\ h^2t \text{ - - - - - - - - - - - (11)}$$

$$I_{\bar{y}}=0.0435\ (c+6h)\ c^2t \text{ - - - - - - - - - - - - - (12)}$$

$$J_m=\frac{\rho}{32.2}\ Jdx \text{ - (13)}$$

In which—

t = thickness of cover in inches.

h = maximum ordinate of section in inches.

c = chord of section in inches.

J = static polar moment of inertia of airfoil shell in inches.[4]

$=I_{\bar{x}}+I_{\bar{y}}$

J_m = mass polar moment of inertia in slug inches squared.

ρ = density in pounds per cubic inch for shell.

L = length of periphery of airfoil section in inches.

$I_{\bar{x}}$ and $I_{\bar{y}}$ are respectively the moments of inertia in inches [4] of the shell about the X and the Y axis through the center of gravity of the section.

The same sections were used as in the case of bending. The mean values of h, t, and c are taken respectively from Figures 9, 10, and 11 and are tabulated in columns 24, 25, and 26. These results are used in the evaluation of A, L, $I_{\bar{x}}$, and $I_{\bar{y}}$ from equations (9), (10), (11), and (12), the results of which are listed in columns 27, 28, 29, and 30. A and L are plotted in Figures 12 and 13.

The mean value of J for any section is tabulated in column 31, and is found from columns 29 and 30 at that section. ΣJ for any section is given in column 32, found by summing all the J's in column 31 between that section and the wing tip. ΣJ for section 19–20 equals $322+1969+2877=5168$ inches.[4]

ΣJ, however, must be multiplied by a factor 3.22 since J is composed of not only the shell alone, but also the ribs and spars. It was assumed that the proportion of J contributed by the ribs, spars, and shell was the same throughout the wing. By comparing the total moment of inertia of a section with that of the wing covering alone, the factor 3.22 was determined for the section between stations 9 and 10.

Thus, for ribs—

$$I_{\text{major}}=0.0418\ c^3h \text{ - - - - - - - - - - - - (14)}$$

$$I_{\text{minor}}=0.454\ ch^3 \text{ - - - - - - - - - - - - - (15)}$$

(See Air Corps Information Circular No. 597, Volume VI.)

I_{major} and I_{minor} are, respectively, the moments of inertia of the rib about the major and minor axis through the center of gravity.

$$I_{\text{major}}=0.0418\,[132-(2\times0.070)]^3\times[22.9-(2\times.070)].$$

$$=0.0418\ (131.86)^3\ 22.76=2{,}170{,}000 \text{ inches.}^4$$

$$I_{\text{minor}}=0.454\times131.86\ (22.76).^3$$

$$=704{,}000 \text{ inches.}^4$$

J for rib $=2{,}170{,}000+704{,}000=2{,}874{,}000$ inches.[4]

For each section,

J for rib $=2{,}874{,}000\times\frac{14}{22}=1{,}825{,}000$ inches,[4] since there are 14 ribs in 22 sections. From equation (13)

J_m for ribs $=\frac{\rho}{32.2}\ 1{,}825{,}000\ dx$ slug inches,[2] in which—

ρ = density of spruce in pounds per cubic inch. $=0.0156$ lbs. per cubic inch.

dx = width of rib $=0.062$ inch.

J_m for ribs $=\frac{0.0156}{32.2}\times1{,}825{,}000\times0.062=55$ slug inches.[2]

In finding the mass polar moment of inertia of the front and rear spars, it was assumed that they were concentrated masses, M_1 and M_2, located on the principal axis distant r_1 and r_2 from the center of gravity of the airfoil section Then—

J_m for spars $=M_1r_1^2+M_2r_2^2$.

Weight of both spars at section 9–10 = 0.7 lbs. per inch span.

Weight of each spar per 20 inches $=0.35\times20=7$ lbs.

$r_1=43.7$ inches; $M_1=\frac{7}{32.2}$.

$r_2=19.3$ inches; $M_2=\frac{7}{32.2}$.

hence—

$$J_m \text{ for spars}=\frac{7}{32.2}\times(43.7)^2+\frac{7}{32.2}\times(19.3)^2$$

$$=415+81=496 \text{ slug inches squared.}$$

This value of J_m is not strictly correct as the weight of the spars is not equally divided according to the assumption. The proportion of front and rear spar weights is given in Appendix A by Mr. G. A. Zink.

$$J_m \text{ for shell}=\frac{\rho}{32.2}\ J\ dx$$

in which—

$J=15{,}642$ inches^4. (See column 31, sec. 9–10.)

ρ = density 3-ply birch = 0.0257 pound per cubic inch.

dx = length of shell = 20 inches.

$$J_m \text{ for shell}=\frac{.0255}{32.2}\times 15642\times 20.$$

$$=248 \text{ slug inches}^2 \text{ at section 9–10.}$$

The total mass polar moment of inertia at section 9–10 is the sum of—

Shell = 248 slug inches^2.
Ribs = 55 slug inches^2.
Spars = 496 slug inches^2.

Total J_m for section 9–10 = 799 slug inches^2.

$$\text{Ratio of } \frac{J_m \text{ of total}}{J_m \text{ of shell}}=\frac{799}{248}=3.22$$

which, when multiplied by the static polar moment of inertia, J, gives the total static polar moment of inertia at that section. Thus, equation (13) becomes

$$J_m=J\cdot 3.22\cdot\frac{\rho}{32.2}\ dx \qquad (16)$$

In the actual computations, it is not necessary to change J to J_m for each section by equation (16) since ρ, the density of plywood covering, is a constant in ordinary airplane construction. Equation (7) can be modified to use J, which simplifies the computations. From equations (8) and (16)—

$$\theta_o=\frac{\Sigma JLdx}{4A^2tE_s}\cdot 3.22\cdot\frac{\rho}{32.2}\ dx$$

$$\theta_o=\frac{\Sigma JL}{4A^2t}\ \frac{1}{E_s}\times 3.22\times\frac{\rho}{32.2}\ dx^2$$

$$\theta_o=\theta\ \frac{1}{E}\times 3.22\times\frac{\rho}{32.2}\ dx^2 \qquad (17)$$

where—

$$\theta_o=\frac{\Sigma JL}{4A^2t} \qquad (18)$$

ΣJ is taken over the part of the wing between the tip and the section considered since the formula for θ_o was derived on the assumption that J is equal to the distributed torque. If J is used instead of J_m, equation (7) becomes

$$f_t=\frac{1}{2\pi}\sqrt{E_s}\sqrt{\frac{\Sigma J\theta}{\Sigma J\theta^2}}\sqrt{\frac{32.2}{3.22\,\rho\,dx^2}} \qquad (19)$$

where ρ = density of plywood covering in pounds per cubic inch.

The average values of $\Delta\theta$ are found from columns 25, 27, 28, and 32 which are listed in column 33. For example—

$$\Delta\theta \text{ at section 19–20}=\frac{\Sigma JL}{4A^2t}\ \frac{5165\times 192}{4\times 640^2\times 0.047}=12.87$$

The $\Delta\theta$'s are proportional to the increments of twist in length dx of the wing. $\Sigma\Delta\theta$ in column 34 is the sum of $\Delta\theta$'s in column 33, starting with zero at the root, since the wing there is considered rigidly fixed. $\Sigma\Delta\theta$ at any station gives the total angle of twist, θ_o in radians at that point when multiplied by $\frac{3.22\,\rho\,dx^2}{32.2\,E_s}$, in which case dx is 20 inches, E_s equals 685,500, and ρ equals 0.0257. (See equations (17) and (18).) $\Sigma\Delta\theta$ plotted in Fig. 14 shows the shape of the θ_o curve. $\Sigma J\times\Sigma\Delta\theta$ and $\Sigma J\times\Sigma\Delta\theta^2$ are found from columns 32, 34, and 35, and are tabulated in 36 and 37 the sum of which columns are used in the substitution of equation (19).

$$f_t=\frac{1}{2\pi}\sqrt{E_s}\sqrt{\frac{\Sigma J\theta}{\Sigma J\theta^2}}\sqrt{\frac{32.2}{1.82\rho dx^2}}$$

$\Sigma J\theta=293{,}343\times 10^3$.
$\Sigma J\theta^2=5182\times 10^7$.
$\rho=0.0255$ lbs. per cubic inch.
$dx=20$ inches.

From tests by the Materials Branch on shear of birch plywood, page 24,

$E_s=685{,}500$ pounds per square inch.

$$f_t=\frac{1}{2\pi}\sqrt{685{,}500}\sqrt{\frac{293{,}343\times 10^3}{5182\times 10^7}}\sqrt{\frac{32.2}{3.22\times 0.0255\times 20^2}}$$

$$=\frac{1}{2\pi}\times 829\times\frac{75.3}{10^2}\times 0.99.$$

$=9.83$ cycles per second.

Experimental $f_t=12$ cycles per second.

$$\text{Per cent error}=\frac{2.17}{12}=18.1 \text{ per cent.}$$

The calculated frequency of torsional vibration is affected by the rigidity of the spars and the modulus of shear of birch plywood. The rigidity of the spars has been neglected, which, if considered, would give a higher calculated frequency. Little testing has been carried out to determine the modulus of shear of birch plywood, and until more extensive tests are made the modulus of shear must be accepted as given.

SHEAR TESTS ON BIRCH PLYWOOD

(Copy of Data Furnished by Materials Branch)

1. The values for the shear test parallel to the face grain on three equal ply, all birch plywood of 1/16 inch thickness for the beam of the spruce flanges are given as follows:

Web No.	Ultimate shear	Modulus of elasticity in shear	Remarks
	Lbs. per sq. in.	*Lbs. per sq. in.*	
1	2,140	715,000	This web, located on upper face at east end, failed first.
2	2,606	656,000	This web, upper face at west end, failed second.
Average	2,480	685,500	

2. The results were obtained from tests on specimens 3×5¼ inches with a gap of ¼ inch between the shear tools.

3. The same average values for modulus of elasticity and shear may be used for the web of beams with plywood flanges. The values for the modulus of elasticity in shear as given above are only approximate inasmuch as the number of specimens tested is limited and accurate values are difficult to obtain.

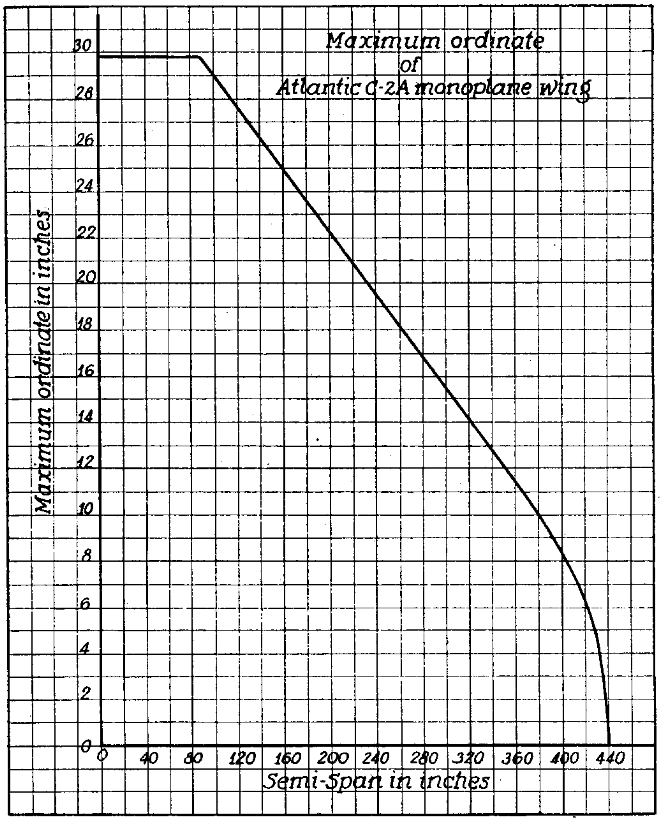

FIGURE 9

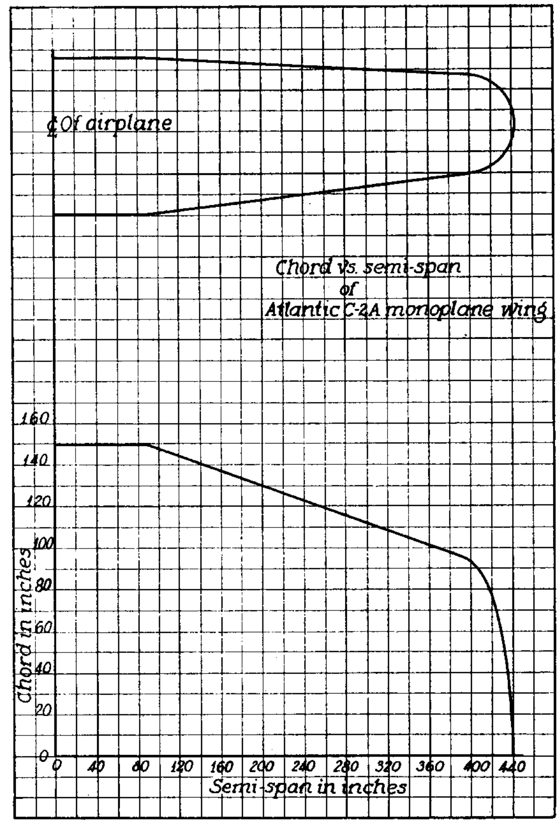

FIGURE 11

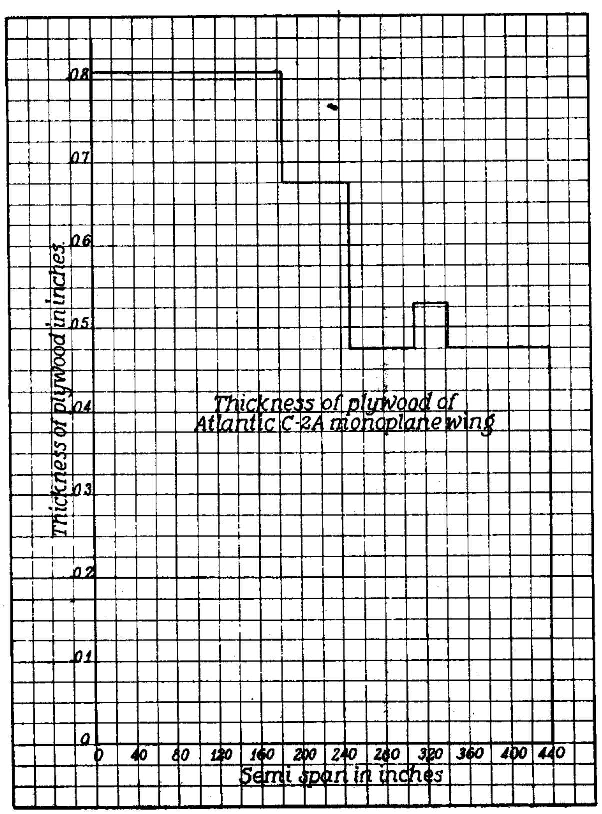

FIGURE 10

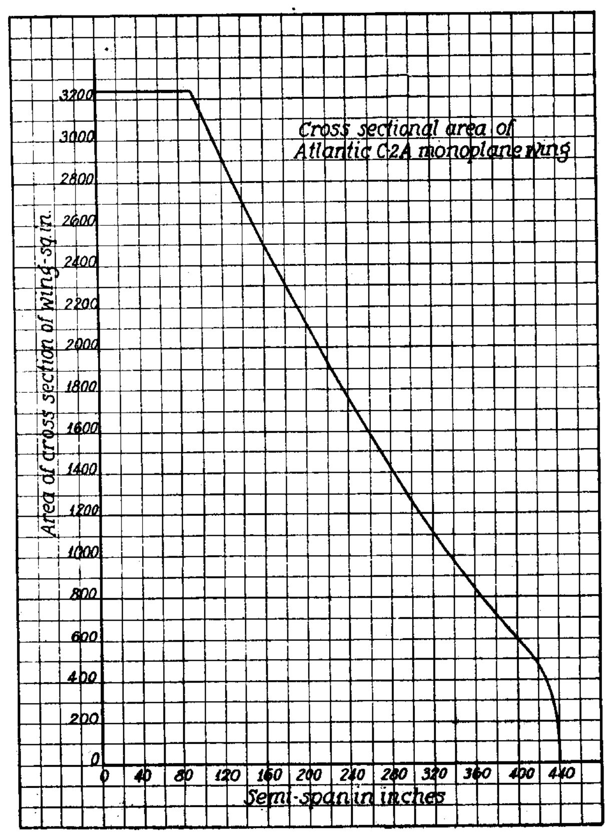

FIGURE 12

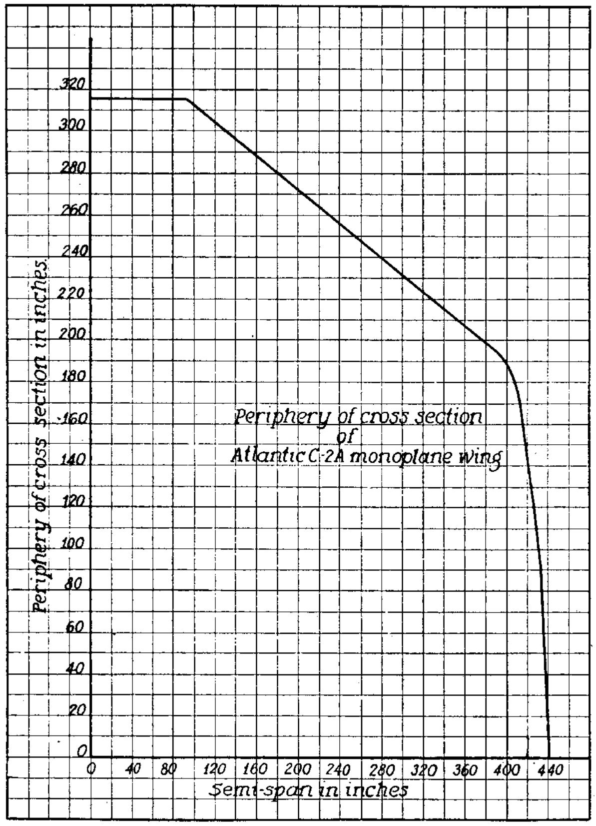

FIGURE 13

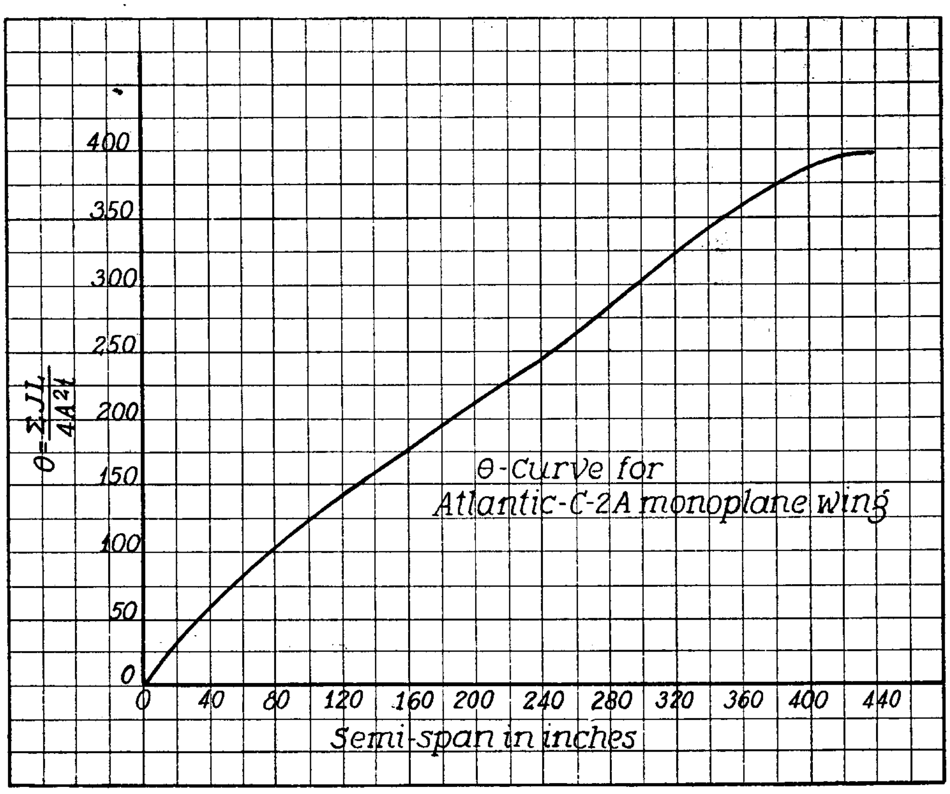

FIGURE 14

22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
Station	dx	h Max. ord.	t=thickness of plywood	C Mean chord	A Airfoil area	L Periphery	I_x	$I_{\bar{y}}$	$J=I_{\bar{x}}+I_{\bar{y}}$	ΣJ	$\Delta\Theta=\frac{\Sigma JL}{4A^2t}$	$\Sigma\Delta\Theta$	$\Sigma\Delta\Theta^2$	$\Sigma J\times\Sigma\Delta\Theta$	$\Sigma J\times\Sigma\Delta\Theta^2$
Tip	*Inches* 17	*Inches* 4.3	*Inches* 0.047	*Inches* 46	*Sq. in.* 320	*Inches* 93	*Inches*[4] 11.25	*Inches*[4] 311	*Inches*[4] 322	322	1.57	397	157×10^3	127.5×10^3	50.5×10^6
21	20	7.4	.047	85	510	172	58.6	1,910	1,969	2,291	8.1	395	156	905	358
20	20	9.3	.047	95	640	192	102	2,775	2,877	5,168	12.9	387	150	2,000	775
19	20	10.6	.047	99	760	201	133	3,260	3,393	8,561	15.9	374	140	3,200	1,200
18	20	12.0	.047	103	880	210	188	3,690	3,878	12,439	18.0	358	128	4,460	1,590
17	20	13.4	.053	106	1,020	217	275	4,560	4,835	17,274	17.4	340	116	5,870	2,000
16	20	14.7	.050	110	1,160	226	324	5,220	5,544	22,818	19.2	322	102	7,350	2,330
15	20	16.1	.047	114	1,320	234	379	5,590	5,969	28,787	20.6	303	92	8,730	2,650
14	20	17.5	.047	117	1,480	241	462	6,220	6,682	35,469	20.8	282.4	79.9	10,050	2,835
13	20	18.8	.0572	121	1,650	250	650	8,500	9,150	44,619	17.9	261.6	68.1	11,650	3,040
12	20	20.2	.0675	124	1,840	257	933	11,100	12,033	56,652	16.0	243.7	59.1	13,800	3,350
11	20	21.5	.0675	128	2,020	266	1,105	12,400	13,505	70,157	16.9	227.7	51.8	16,000	3,640
10	20	22.9	.070	132	2,200	275	1,342	14,300	15,642	85,799	17.4	210.8	44.2	18,050	3,780
9	20	24.2	.081	135	2,390	282	1,780	18,000	19,780	105,579	16.1	193.4	37.4	20,400	3,940
8	20	25.6	.081	139	2,590	291	2,050	20,000	22,050	127,629	17.1	177.3	31.4	22,600	4,000
7	20	27.0	.081	142	2,790	298	2,340	21,600	23,940	151,569	17.9	160.2	25.6	24,200	3,880
6	20	28.3	.081	146	3,020	306	2,645	23,700	26,345	177,914	18.4	142.3	20.3	25,300	3,610
5	20	29.5	.081	149	3,210	314	2,930	25,500	28,430	206,344	19.4	123.9	15.3	25,600	3,160
4	20	29.8	.081	150	3,230	316	3,020	26,100	29,120	235,464	22.0	104.5	10.9	24,600	2,560
3	20	29.8	.081	150	3,230	316	3,020	26,100	29,120	264,584	24.8	82.5	6.8	21,800	1,800
2	20	29.8	.081	150	3,230	316	3,020	26,100	29,120	293,704	27.5	57.7	3.34	16,900	980
1	20	29.8	.081	150	3,230	316	3,020	26,100	29,120	322,824	30.2	30.2	.91	9,750	294
Root													Sum$\Sigma J\times\Sigma\Delta\Theta=293,343\times10^3$		Sum$\Sigma J\times\Sigma\Delta\Theta^2=51,823\times10^6$

APPENDIX A

The weights of the rear and front spars, as calculated from the details of the spars between stations 9 and 10, are 5.84 and 7.08 pounds respectively. These weights were based on a density of 0.0156 pounds per cubic inch for the spar material.

The dimensions referred to are at a distance of 190.0 inches from the fuselage, and are as follows:

c (chord) = 132.0 inches.

h (maximum ordinate) = 22.9 inches.

t (thickness of plywood shell) = 0.070 inches.

The formulas used in the following computations are the same as those used in the discussion. A sample computation for the determining of J_m for the wing cell follows:

$$I_x = [0.119\ (22.9) + 0.256\ (132.0)](22.9)^2\ (0.070)$$
$$= 1{,}342 \text{ inches}^4.$$
$$I_y = (0.0435)[132.0 + 6\ (22.9)](132)^2\ (0.070)$$
$$= 14{,}300 \text{ inches}^4.$$
$$J = I_x + I_y$$
$$= 15{,}642 \text{ inches}^4.$$
$$J_m = \frac{(0.0255)}{32.2}\ (20)\ (15{,}642)$$
$$= 248.0 \text{ pounds-inches}^2.$$

J_m for the ribs is found as follows:

$c_1 = c\text{-}2\ (0.070)$; $c_1 = 131.86$ inches.

$h_1 = \text{h-}2\ (0.070)$; $h_1 = 22.76$ inches.

Center distance of spars = 63.0 inches.

$$I_y = 0.0418\ c_1^3 h_1.$$
$$I_x = 0.454\ c_1 h_1^3.$$
$$I_y = 0.0418\ (131.86)^3\ (22.76)$$
$$= 2{,}170{,}000 \text{ inches}^4.$$
$$I_x = 0.454\ (131.86)\ (22.76)^3$$
$$= 704{,}000 \text{ inches}^4.$$
$$J = I_x + I_y.$$
$$= 2{,}874{,}000 \text{ inches}^4.$$
$$J_m = \frac{(0.0156)}{32.2}\ (0.062)\ (2{,}874{,}000)\left(\frac{14}{22}\right)$$
$$= 55.0 \text{ pounds-inches}^2.$$

J_m for the spars.

Distance of C. G. from leading edge of wing = 62.7 inches.

$r_1 = 19.3$ inches, $r_2 = 43.7$ inches.

$$M_1 = \frac{5.84}{32.2},\ M_2 = \frac{7.08}{32.2}$$
$$J_m = \frac{5.84}{32.2}\ (19.3)^2 + \frac{7.08}{32.2}\ (43.7)^2$$
$$= 487.5 \text{ pounds-inches}^2.$$

Weight of rear spar/20 inches of length, stations 9–10.

2.92 pounds weight of web.

2.92 pounds weight of beam.

Total weight of spar, rear, 20 inches length = 5.84 pounds.

Weight of front spar/20 inches of length, stations 9–10.

4.21 pounds weight of web.

2.87 pounds weight of beam.

Total weight of spar, front, 20 inches length = 7.08 pounds.

C. G. along x axis = 132.0 (0.475) = 62.7 inches.

C. G. along y axis = 22.9 (0.37) = 8.47 inches.

Shell—

$$J = I_x + I_y.$$
$$I_x = (0.119h + 0.256c)h^2 t.$$
$$I_x = [0.119\ (22.9) + 0.256\ (132.0)](22.9)^2\ (0.070).$$
$$I_x = 1{,}342.$$
$$I_y = 14{,}300.$$
$$J = 15{,}642.$$
$$J_m = \frac{0.0255}{32.2}\ (20)\ (15{,}642) = 248.0.$$

Rib—

$$I_y = 0.0418\ c_1^3 h_1.$$
$$I_x = 0.454\ c_1 h_1^3.$$

$c_1 = c\text{-}2\ (0.070) = 131.86$ inches.

$h_1 = \text{h-}2\ (0.070) = 22.76$ inches.

C. L. rear spar to C. L. front = 63.0 inches.

The total J_m for section considered is equal to the sum of the J_m's of the shell, rib, and spars, or—

$$J_m = 487.5 + 55.0 + 248.0$$
$$= 790.5 \text{ pounds-inches}^2.$$
$$\text{Ratio of } \frac{J_m \text{ of total}}{J_m \text{ of shell}} = \frac{790.5}{248.0} = 3.19.$$